

Cops and Robber Game with a Fast Robber on Interval, Chordal, and Planar Graphs

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Abstract

We consider a variant of the Cops and Robber game, introduced by Fomin, Golovach, Kratochvíl, in which the robber has unbounded speed, i.e. can take any path from her vertex in her turn, but she is not allowed to pass through a vertex occupied by a cop. We study this game on interval graphs, chordal graphs, planar graphs, and hypercube graphs. Let $c_\infty(G)$ denote the number of cops needed to capture the robber in graph G in this variant. We show that if G is an interval graph, then $c_\infty(G) = O(\sqrt{|V(G)|})$, and we give a polynomial-time 3-approximation algorithm for finding $c_\infty(G)$ in interval graphs. We prove that for every n there exists an n -vertex chordal graph G with $c_\infty(G) = \Omega(n/\log n)$. Let $tw(G)$ and $\Delta(G)$ denote the treewidth and the maximum degree of G , respectively. We prove that for every G , $tw(G) + 1 \leq (\Delta(G) + 1)c_\infty(G)$. Using this lower bound for $c_\infty(G)$, we show two things. The first is that if G is a planar graph (or more generally, if G does not have a fixed apex graph as a minor), then $c_\infty(G) = \Theta(tw(G))$. This immediately leads to an $O(1)$ -approximation algorithm for computing c_∞ for planar graphs. The second is that if G is the m -hypercube graph, then there exist constants $\eta_1, \eta_2 > 0$ such that $\eta_1 2^m / (m\sqrt{m}) \leq c_\infty(G) \leq \eta_2 2^m / m$.

Keywords: Cops and Robber game, Treewidth, Interval and chordal graphs, Planar graphs

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1 Introduction

The game of *Cops and Robber* is a perfect information game, played in a graph G . The players are a set of cops and a robber. Initially, the cops are placed at vertices of their choice in G (where more than one cop can be placed at a vertex). Then the robber, being fully aware of the cops' placement, positions herself at one of the vertices of G . Then the cops and the robber move in alternate rounds, with the cops moving first; however, players are permitted to remain stationary in their turn if they wish. The players use the edges of G to move from vertex to vertex. The cops win, and the game ends, if eventually a cop moves to the vertex currently occupied by the robber; otherwise, i.e. if the robber can elude the cops forever, the robber wins.

This game was defined (for one cop) by Winkler and Nowakowski [22] and Quilliot [24], and has been studied extensively. For a survey of results on this game, see the survey by Hahn [15]. The famous open question in this area is Meyniel's conjecture, published by Frankl [13], which states that for every connected graph on n vertices, $O(\sqrt{n})$ cops are sufficient to capture the robber. The best result so far is that

$$n2^{-(1-o(1))\sqrt{\log_2 n}}$$

cops are sufficient to capture the robber. This was proved independently by Lu and Peng [20], and Scott and Sudakov [25].

One interesting fact about the Cops and Robber game is that, many scholars have studied the game, and yet it is not really well understood: although the upper bound $O(\sqrt{n})$ was conjectured in 1987, no upper bound better than $n^{1-o(1)}$ has been proved since then. As an another example, no efficient approximation algorithm for finding the number of cops needed to capture the robber in a given graph has been developed.

One might try to change the rules of the game a little in order to get a more approachable problem, and/or to understand what property of the original game causes the difficulty. Several variations of the game has been studied, by changing the rules slightly, e.g. by limiting the visibility of the cops [5], by limiting the visibility of both players [17], by changing the definition of capturing [3], or by allowing the players to move only in a certain direction along each edge [14].

The approach chosen by Fomin, Golovach, Kratochvíl, Nisse, and Suchan [12] is to allow the robber move faster than the cops. Inspired by their work, in this paper we let the robber take *any path* from her current position in her turn, but she is not allowed to pass through a vertex occupied by a cop. The parameter of interest is the *cop number* of

G , which is defined as the minimum number of cops needed to ensure that the cops can win. We denote the cop number of G by $c_\infty(G)$, in which the ∞ at the subscript indicates that the robber has unbounded speed. A nice fact about this variation is its analogy with the so-called Helicopter Cops and Robber game (defined in [26], see Section 4 for the definition). This is a real-time pursuit-evasion game with a robber of unbounded speed, for which Seymour and Thomas have shown that the number of cops needed equals the treewidth of the graph (which is a fairly well understood parameter) plus one [26]. Thus one may hope to get good bounds for the cop number in terms of treewidth by relating our variant of the Cops and Robber game and the Helicopter Cops and Robber game, and this is what we do in Section 4. However, one should not be deceived by this analogy; the cop number can be arbitrarily smaller than the treewidth: any graph with small domination number and large treewidth (e.g., a complete graph) is such an example. Therefore, this paper can also be regarded as an attempt to find connections between the original Cops and Robber game and the Helicopter Cops and Robber game by studying an in-between game. Nevertheless, treewidth is closely connected with cop number, and in Section 4, which is completely devoted to this connection, we prove bounds for cop number in terms of treewidth. In Sections 5, 6 and 7, we will see three applications of these bounds. Tree and path decompositions arise naturally and are important when studying the cop number, and the idea of several proofs in Sections 2 and 3 is based on them (although they do not appear explicitly in these sections).

This variant was first studied by Fomin, Golovach, Kratochvíl [11]. They proved that computing $c_\infty(G)$ is an NP-hard problem, even if G is a split graph. (A *split graph* is a graph whose vertex set can be partitioned into a clique and an independent set.) This variant was further studied by Frieze, Krivelevich and Loh [14], where the authors' approach is based on expansion. In [14], it is shown that for each n , there exists a connected graph on n vertices with cop number $\Theta(n)$.

Let G be a connected graph with n vertices. In Section 2, we show that if G is an interval graph, then $c_\infty(G) = O(\sqrt{n})$ and provide examples for which this bound is tight. We also give a polynomial-time 3-approximation algorithm for finding $c_\infty(G)$. In Section 3, we prove that for every n there exists a chordal graph G with $c_\infty(G) = \Omega(n/\log n)$. Let $tw(G)$ and $\Delta(G)$ denote the treewidth and the maximum degree of G , respectively. In Section 4, we prove that for every G ,

$$\frac{tw(G) + 1}{\Delta(G) + 1} \leq c_\infty(G) \leq tw(G) + 1,$$

and provide examples for which these bounds are tight. We will see applications of this

result in the three subsequent sections. In Section 5, we show that if G is a planar graph (or more generally, if G does not have a fixed apex graph H as a minor), then $c_\infty(G) = \Theta(tw(G))$. This immediately leads to an $O(1)$ -approximation algorithm for computing the cop number of planar (in general, apex-minor-free) graphs. In Section 6, we show that if G is the Cartesian product of m copies of K_k , then there exist positive constants κ_1, κ_2 such that

$$\frac{\kappa_1 n}{km\sqrt{m}} \leq c_\infty(G) \leq \min \left\{ \frac{n}{k}, \frac{\kappa_2 n}{\sqrt{m}} \right\}.$$

Moreover, if G is the m -hypercube graph, then there exist constants $\eta_1, \eta_2 > 0$ such that

$$\frac{\eta_1 n}{m\sqrt{m}} \leq c_\infty(G) \leq \frac{\eta_2 n}{m}.$$

In Section 7 we give a short proof for the fact that for each n , there exists a connected graph on n vertices with cop number $\Theta(n)$, which is proved in [14] using other ideas. We conclude with some open problems in Section 8.

1.1 Preliminaries and notation

Let G be the graph in which the game is played. In this paper G is always finite, and n always denotes the number of vertices of G . We will assume that G is simple, because deleting multiple edges or loops does not affect the set of possible moves of the players. We consider only connected graphs, since the cop number of a disconnected graph obviously equals the sum of the cop numbers for each connected component. As we are only interested in studying the cop number, we may assume without loss of generality that the cops choose vertices of our choice in the beginning, since they can move to the vertices of their choice later.

For a subset A of vertices, the *neighbourhood* of A , written $N(A)$, is the set of vertices that have a neighbour in A , and the *closed neighbourhood* of A , written $\overline{N}(A)$, is the union $A \cup N(A)$. If $A = \{v\}$ then we may write $N(v)$ and $\overline{N}(v)$ instead of $N(A)$ and $\overline{N}(A)$, respectively. A *dominating set* is a subset A of vertices with $V(G) = \overline{N}(A)$, and the *domination number* of G is the minimum size of a dominating set of G . The subgraph induced by A is written $G[A]$, and the subgraph induced by $V(G) - A$ is written $G - A$.

2 Interval Graphs

Graph G is an *interval graph* if there is a correspondence between its vertices and a set of closed intervals on the real line, such that two vertices are adjacent in G if and only if their corresponding intervals intersect. Let G be an interval graph. Fomin et al. [11] proved that if the robber has fixed speed, the number of cops needed to capture the robber can be computed in polynomial time. The complexity of computing $c_\infty(G)$ was left open in [11]. As a partial answer, in this section we prove that this problem is 3-approximable. We also prove that $c_\infty(G) = O(\sqrt{n})$ for all connected interval graphs G , and provide examples for which this bound is tight.

Definition (k -wide). For a subgraph H of G , say H is k -wide if

- (i) H is k -connected, and
- (ii) for any $S \subseteq V(G)$ with $|S| < k$ we have $V(H) \not\subseteq \overline{N}(S)$.

Lemma 2.1. *If G has a k -wide subgraph H then $c_\infty(G) \geq k$.*

Proof. Say a cop *controls* a vertex u if the cop is at u or at an adjacent vertex. Suppose that there are less than k cops in the game, and they initially start at a subset S of vertices. By condition (ii), there is a vertex $v \in V(H) \setminus \overline{N}(S)$, i.e. v is controlled by no cop. The robber starts at v , and will always remain in H . After each move of the cops, the set of vertices occupied by them has size less than k . Hence by condition (ii), there exists a vertex x of H that is not controlled by any of the cops. By condition (i), H is k -connected, so as the robber is currently in H , and the number of cops is less than k , there is a cop-free path to x . The robber moves there and will not be captured in the next round. Since she can elude forever by using this strategy, at least k cops are needed to capture her. ■

For the rest of this section, G is a connected interval graph. Consider a set of closed intervals whose intersection graph is G , and denote by I_v the interval corresponding to the vertex $v \in V(G)$. We may assume without loss of generality that none of the intervals have zero length. Such a representation can be found in polynomial time (see [16] for instance). Let $x_1 < x_2 < \dots < x_{l+1}$ be the set of distinct endpoints of the intervals, and let y_1, y_2, \dots, y_l be points satisfying $x_i < y_i < x_{i+1}$ for $1 \leq i \leq l$. Also, define $V_i = \{v \in V(G) : y_i \in I_v\}$ for all $1 \leq i \leq l$. It is clear that each $G[V_i]$ is a clique for

$1 \leq i \leq l$ (recall that $G[V_i]$ denotes the subgraph induced by V_i). Furthermore, $l \leq 2n$ and the sets V_1, \dots, V_l cover the vertices of G .

We say A is a *cut-set* of G if $G - A$ has more connected components than G .

Lemma 2.2. *Every minimal cut-set X of G is one of the V_i 's. Moreover, if $X = V_i$ is a cut-set, then for each $u_1 \in V_{i_1} \setminus X$ and $u_2 \in V_{i_2} \setminus X$ satisfying $i_1 < i < i_2$, u_1 and u_2 lie in different components of $G - X$.*

Proof. For an index $1 \leq i \leq l$, say point y_i is a *cut-point* if there exists a vertex $v \in V(G)$ with both endpoints of I_v lying strictly on the left of y_i , and also a vertex $v' \in V(G)$ with both endpoints of $I_{v'}$ lying strictly on the right of y_i . If y_i is a cut-point then clearly V_i is a cut-set of G .

Now, let X be a minimal cut-set of G . Let u_1, u_2 be vertices in different components of $G - X$, with $I_{u_1} = [x_a, x_b]$, $I_{u_2} = [x_c, x_d]$, and assume by symmetry that $a < b < c < d$. For each i with $b \leq i \leq c - 1$, y_i is a cut-point. If for all of the i 's in this range, there was a vertex $v_i \in V_i \setminus X$, then $u_1 v_b v_{b+1} \dots v_{c-1} u_2$ would be a (u_1, u_2) -path in $G - X$. As such a path does not exist, there is an i in this range such that $V_i \subseteq X$. But then V_i is a cut-set of G , hence $X = V_i$.

For the second statement, let $X = V_i$ be a cut-set, $u_1 \in V_{i_1} \setminus X$ and $u_2 \in V_{i_2} \setminus X$ such that $i_1 < i < i_2$. Let $I_{u_1} = [x_a, x_b]$, $I_{u_2} = [x_c, x_d]$, and so $x_a < x_b < y_i < x_c < x_d$. Every (u_1, u_2) -path contains a vertex whose corresponding interval contains y_i , but all such vertices are in X . Hence there is no (u_1, u_2) -path in $G - X$. ■

Definition ($G[a, b]$, interval subgraph, $w(G)$). We write $G[a, b]$ for the subgraph induced by $\bigcup_{a \leq i \leq b} V_i$ (for $1 \leq a \leq b \leq l$), and we call each of these an *interval subgraph*. Let $w(G)$ be the maximum number M such that G has an M -wide interval subgraph.

Lemma 2.3. *It is possible to compute $w(G)$ in polynomial time.*

Proof. Fix an interval subgraph $G[a, b]$. It is easy to see that there is an $S \subseteq V(G)$ with $V(G[a, b]) \subseteq \overline{N}(S)$ if and only if the domination number of $G[a, b]$ is at most $|S|$, that is, if there is a set of $|S|$ vertices of G dominating the vertices of $G[a, b]$, then there exists such a set inside $G[a, b]$. Moreover, $G[a, b]$ is an interval graph so its domination number can be found in polynomial time (using a greedy algorithm). The connectivity of $G[a, b]$ can also be computed in polynomial time (see [9] for example). Therefore, the largest M such that $G[a, b]$ is M -wide can be computed in polynomial time. Recall that $w(G)$ is the maximum number M such that G has an M -wide interval subgraph. The total number of interval subgraphs is $O(l^2) = O(n^2)$, so $w(G)$ can be computed in polynomial time. ■

The following lemma gives an appropriate upper bound for $c_\infty(G)$.

Lemma 2.4. *We have $c_\infty(G) \leq 3w(G)$.*

Proof. We just need to give a strategy for $3w(G)$ cops to capture the robber. Let $M = w(G)$. There are three teams of cops, each of size M . At the beginning the first team starts at a vertex in V_1 , the second team starts at a vertex in V_l , and the third team starts at an arbitrary vertex. Suppose that the robber starts at a vertex r . The cops' strategy consists of several (at most l) phases, in each of which they reduce the free space of the robber. The following invariant is true at the start of each phase: the j -th team ($j = 1, 2$) is in a subset $X_j \subseteq V_{i_j}$ such that they block the robber from escaping $G[i_1, i_2]$.

Note that during this phase, if the robber goes to a vertex in $V_{i_1} \cup V_{i_2}$ then she will be captured immediately by the first or second team (recall that each $G[V_i]$ is a clique). If $i_2 \leq i_1 + 1$ then she should move to a vertex in $V_{i_1} \cup V_{i_2}$ and will be captured immediately, so assume that $i_2 > i_1 + 1$. Since $G[i_1 + 1, i_2 - 1]$ is not $(M + 1)$ -wide, either $G[i_1 + 1, i_2 - 1]$ has a minimal cut-set X of size at most M , or $G[i_1 + 1, i_2 - 1]$ has a dominating set X of size at most M .

In the second case, the third team moves to X (while the first and second teams stay still and block the robber from escaping $G[i_1, i_2]$), and the robber will be captured in the next round.

In the first case, the third team moves to X , and suppose that $X = V_{i_3}$ (by Lemma 2.2, X is of this form). Suppose that the robber moves to r right after the third team has settled in X and j be an index such that $r \in V_j$. If $j = i_3$ then the third team immediately captures her (since $G[V_{i_3}]$ is a clique), so assume, by symmetry, that $i_1 < j < i_3$. Now, the first team together with the third team block the robber from escaping the subgraph $G[i_1, i_3]$ (by the second statement in Lemma 2.2). The second and third team switch roles and this phase finishes. Note that $i_3 - i_1 < i_2 - i_1$ so the total number of phases is not larger than l . ■

Theorem 2.5. *There exists a polynomial-time 3-approximation algorithm for computing $c_\infty(G)$ when G is an interval graph.*

Proof. Given G , the sequence (V_1, V_2, \dots, V_l) can be found efficiently. Then $w(G)$ can be computed in polynomial time by Lemma 2.3. The value $3w(G)$ is a 3-approximation for $c_\infty(G)$ by Lemmas 2.1 and 2.4. ■

Next we prove that $c_\infty(G) = O(\sqrt{n})$. Before doing so, we note that this bound is tight: let G be the strong product of the path on $3m$ vertices and the complete graph on m vertices. That is,

$$V(G) = \{1, 2, \dots, 3m\} \times \{1, 2, \dots, m\},$$

and

$$\{(i, j), (k, l)\} \in E(G) \text{ if } (i, j) \neq (k, l) \text{ and } |i - k| \leq 1.$$

Then G is an interval graph with $3m^2$ vertices, and is m -wide itself, hence

$$c_\infty(G) \geq m = \Omega(\sqrt{|V(G)|}).$$

We will need a lemma about minimum dominating sets in interval graphs, which may not be the best possible, but suffices for our purposes.

Lemma 2.6. *Let A be a minimum dominating set of G . Every vertex $v \in A$ is adjacent to at most two vertices of A , and every vertex $v \notin A$ is adjacent to at most five vertices of A .*

Proof. Let $I_v = [x, y]$ be the interval corresponding to vertex v . First, let $v \in A$. If there is a vertex $u \in A$ whose corresponding interval contains I_v , then $\overline{N}(v) \subseteq \overline{N}(u)$, which contradicts the minimality of A . If there is a vertex $u \in A$ whose corresponding interval is contained in I_v , then $\overline{N}(u) \subseteq \overline{N}(v)$, which contradicts the minimality of A . So for every $u \in A$ that is adjacent to v , the interval corresponding to u contains exactly one of x and y . If there are two distinct vertices in $N(v) \cap A$ whose corresponding intervals contain x , then one can remove one of them (the one whose left-end-point of the corresponding interval is more to the right) from A , and still have a dominating set, which contradicts the minimality of A . Thus there exists at most one vertex in $N(v) \cap A$ whose corresponding interval contains x . Similarly, there exists at most one vertex in $N(v) \cap A$ whose corresponding interval contains y , so $|N(v) \cap A| \leq 2$.

Second, let $v \notin A$. If there is a vertex $u \in A$ whose corresponding interval contains I_v , then since u is adjacent to at most two vertices of A , v is adjacent to at most three vertices of A . So we may assume that this is not the case. If there are two distinct $u_1, u_2 \in A$ whose corresponding intervals are contained in I_v , then $\overline{N}(u_1) \cup \overline{N}(u_2) \subseteq \overline{N}(v)$, which contradicts the minimality of A . By an argument similar to the one in the previous case, it can be shown that there are at most two distinct vertices in $A \cap N(v)$ whose corresponding intervals contain x . Similarly, there are at most two distinct vertices in $A \cap N(v)$ whose corresponding intervals contain y . Thus, v is adjacent to at most five vertices of A . ■

Theorem 2.7. *Let G be a connected interval graph with n vertices. Then $c_\infty(G) = O(\sqrt{n})$.*

Proof. By Lemma 2.4 it is enough to show that $w(G) = O(\sqrt{n})$. Let $G[a, b]$ be an arbitrary interval subgraph of G . We just need to prove that $G[a, b]$ is not $(\sqrt{5n} + 3)$ -wide. Choose two arbitrary vertices $u_a \in V_a, u_b \in V_b$. Let a' be the smallest index in $\{a, a + 1, \dots, b\}$ with $u_a \notin V_{a'}$, and b' be the largest index in $\{a, \dots, b\}$ with $u_b \notin V_{b'}$. If either of these indices does not exist or $a' > b'$, then $\{u_a, u_b\}$ is a dominating set for $G[a, b]$, so it is not $(\sqrt{5n} + 3)$ -wide. So, we may assume that a' and b' exist.

Consider the graph $G[a', b']$. Let n_1 be its number of vertices, T be a minimum dominating set for it, and δ be its minimum degree. Let $t = |T|$. Note that $T \cup \{u_a, u_b\}$ is a dominating set for $G[a, b]$, so the domination number of $G[a, b]$ is at most $t + 2$. Moreover, $G[a', b']$ is an interval graph, so by Lemma 2.6, every vertex $v \in V(G[a', b']) \setminus T$ is adjacent to at most five vertices of T , and every vertex $v \in T$ is adjacent to at most two vertices of T , hence (denoting the degree of u in $G[a', b']$ by $\deg(u)$) we have

$$t(\delta + 1) \leq \sum_{u \in T} (\deg(u) + 1) \leq 5n_1 \leq 5n,$$

so $\min\{t, \delta + 1\} \leq \sqrt{5n}$.

If $t \leq \sqrt{5n}$ then the domination number of $G[a, b]$ is at most $\sqrt{5n} + 2$ so it is not $(\sqrt{5n} + 3)$ -wide. So we may assume that $\delta + 1 \leq \sqrt{5n}$. Let u be a vertex of minimum degree in $G[a', b']$, which is contained in some $V_i, a' \leq i \leq b'$. Thus $|V_i| \leq \delta + 1 \leq \sqrt{5n}$ and V_i is a cut-set in $G[a, b]$ (as it separates u_a, u_b), so $G[a, b]$ is not $(\sqrt{5n} + 1)$ -connected, and hence not $(\sqrt{5n} + 3)$ -wide. ■

3 Chordal Graphs

A *chordal* graph is a graph that does not have an induced cycle with more than 3 vertices. Note that any interval graph is chordal. It is well known that in the original Cops and Robber game, a single cop can capture the robber in a chordal graph (an easy way to see this is by considering a tree decomposition of G in which each bag induces a clique in G). However, when the robber has unbounded speed the situation is quite different. In this section we prove that there exist chordal graphs G with $c_\infty(G) = \Omega(n / \log n)$. More precisely, it is shown that for every positive integer m , there exists a chordal graph G with $O(m \log m)$ vertices having $c_\infty(G) \geq m$.



Figure 1: Examples of accessible pairs

Definition (access, accessible). Say the robber *has access* to a subset $X \subseteq V(G)$ if there exists a cop-free path from the robber's vertex to a vertex in X . A pair (X, v) with $X \subseteq V(G)$ and $v \in V(G)$ is called *accessible* if

- $c_\infty(G) \geq |X|$,
- $N(v) = X$, and
- if there are $|X| - 1$ cops in the game, then there exists a strategy for the robber with the following properties: the robber has access to X in every round, but she never moves to a vertex in $X \cup \{v\}$.

In Figure 1, (X_i, v_i) is an accessible pair in G_i for $i = 1, 2$.

Lemma 3.1. *Let G_1, G_2 be graphs on disjoint vertex sets, and for $i = 1, 2$, (X_i, v_i) be an accessible pair in G_i with $|X_i| = k$. Let G be a graph with vertex set $V(G) = V_1 \cup U_1 \cup X \cup U_2 \cup V_2 \cup \{v\}$, and such that*

- *For $i = 1, 2$, $V_i = V(G_i) \setminus \{v_i\}$.*
- *We have $|U_1| = |X| = |U_2| = 2|X_1| = 2|X_2| = 2k$ and V_1, U_1, X, U_2, V_2 are disjoint.*
- *The following pairs induce complete bipartite subgraphs of G : (X_1, U_1) , (U_1, X) , (X, U_2) , (U_2, X_2) .*
- *There is no other edge between any two of V_1, U_1, X, U_2, V_2 , but there can be arbitrary edges inside U_1, X, U_2 .*
- *The set of neighbours of v is precisely X .*

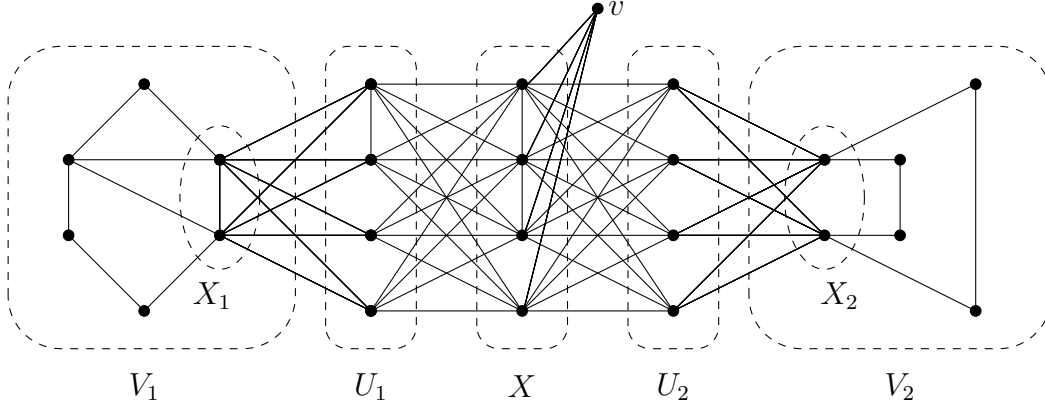


Figure 2: An example for Lemma 3.1

Then (X, v) is an accessible pair in G .

In Figure 2 an example of such a G is given, where G_1 and G_2 are graphs shown in Figure 1.

Proof. Assume that there are $2k - 1$ cops in the game. We prove that the robber has an escaping strategy that evades the cops forever, and is such that she has access to X in every round, but never moves to $X \cup \{v\}$. Let $A_i = V_i \cup U_i$ for $i = 1, 2$. The strategy has the following invariant: at the end of each round, the robber is at a vertex of V_j for some $1 \leq j \leq 2$, such that there are less than k cops in A_j , and the robber has access to X_j . If we provide such a strategy, then since the robber has access to X_j and there are k disjoint paths from X_j to X , the robber has access to X in every round. We may assume without loss of generality that all the cops start at some vertex in V_2 , and the robber starts at some vertex in V_1 , so the invariant holds at the beginning (with $j = 1$).

Assume that the invariant holds at the end of the previous round, say with $j = 1$. This means that the robber is at a vertex of V_1 , has access to X_1 , and there are less than k cops in A_1 . In the next round, first the cops move. If after their move, there are still less than k cops in A_1 , then the robber assumes the game is actually played in G_1 , where she considers all cops in V_1 as they are in G_1 , and she considers all cops in $V(G) \setminus V_1$ as if they are at v_1 ; then she just plays her escaping strategy in G_1 , thus she will not go to $X_1 \cup \{v_1\}$ and will not be captured in the next round. Recall that v_1 is the vertex in G_1 whose set of neighbours is X_1 .

In the other case, there are at least k cops in A_1 . There are at most $k - 1$ cops in A_2

at this moment, and in particular, at most $k - 1$ cops in V_2 . Recall that (X_2, v_2) is an accessible pair in G_2 , which means, in particular, that there exists a vertex $u \in V_2$ such that at this moment there is a cop-free path P from X_2 to u . (To see this, note that if one just considers the graph induced by V_2 and assumes that the game is played only in this subgraph, then the robber can choose a vertex that has access to X_2 .) Since at the end of the previous round there were less than k cops in A_1 , there are less than k cops in V_1 at this moment. Hence the robber has access to X_1 (note that cops in $V(G) \setminus V_1$ will not block the robber's access to X_1), through which she can pass through U_1, X, U_2, X_2 (notice that each of these has at least one cop-free vertex), and finally go to u along the path P . ■

It is easy to verify that if both G_1 and G_2 are chordal graphs and the subgraphs induced by U_1, X , and U_2 are complete graphs, then the resulting graph G is chordal as well. This lets us deduce the following lower bound.

Theorem 3.2. *For every positive integer m , there exists a chordal graph G having $O(m \log m)$ vertices and $c_\infty(G) \geq m$.*

Proof. For every m , let $g(m)$ denote the number of vertices of the smallest connected chordal graph that has an accessible pair (X, v) with $|X| = m$. Then, by Lemma 3.1 and the discussion above,

$$g(2) \leq 7, \quad g(m) \leq 2(g(\lceil m/2 \rceil) - 1) + 6\lceil m/2 \rceil + 1,$$

which gives $g(m) = O(m \log m)$. ■

4 Cop Number and Treewidth

A *tree decomposition* of a graph G is a pair (T, W) , where T is a tree and $W = (W_t : t \in V(T))$ is a family of subsets of $V(G)$ such that

- (i) $\bigcup_{t \in V(T)} W_t = V(G)$, and every edge of G has both endpoints in some W_t , and
- (ii) For every $v \in V(G)$, the set $\{t : v \in W_t\}$ induces a subtree of T .

The *width* of (T, W) is

$$\max\{|W_t| - 1 : t \in V(T)\},$$

and the *treewidth* of G , written $tw(G)$, is the minimum width of a tree decomposition of G .

We will use the following facts about tree decompositions, whose proofs can be found in Section 12.3 of the textbook by Diestel [8].

Proposition 4.1. *Let (T, W) be a tree decomposition of a graph G .*

- (a) *Let A be the vertex set of a clique in G . Then there is a $t \in V(T)$ with $A \subseteq W_t$.*
- (b) *Let $t_1 t_2$ be an edge of T , and let T_1 and T_2 be the components of $T - t_1 t_2$, with $t_1 \in T_1$ and $t_2 \in T_2$. Define $X = W_{t_1} \cap W_{t_2}$, $U_1 = \cup_{t \in T_1} W_t$ and $U_2 = \cup_{t \in T_2} W_t$. Then X is a cut-set in G , and there is no edge between $U_1 \setminus X$ and $U_2 \setminus X$.*

For the original Cops and Robber game, Joret, Kamiński, and Theis [18] proved that for every G , $\frac{tw(G)}{2} + 1$ cops are sufficient to capture the robber.

Write $\Delta = \Delta(G)$ for the maximum degree in G . In this section we prove that for every G ,

$$\frac{tw(G) + 1}{\Delta(G) + 1} \leq c_\infty(G) \leq tw(G) + 1.$$

Moreover, we prove that these bounds are tight. To prove the lower bound, we relate our Cops and Robber game with another pursuit-evasion game, called the Helicopter Cops and Robber game. This game, introduced by Seymour and Thomas [26], has two different versions, and the one we define here is called jump-searching.

Definition (Helicopter Cops and Robber game (the jump-searching version)). For $X \subseteq V(G)$, an X -*flap* is the vertex set of a connected component of $G - X$. Two subsets $X, Y \subseteq V(G)$ *touch* if $\overline{N}(X) \cap Y \neq \emptyset$. A *position* is a pair (X, R) , where $X \subseteq V(G)$ and R is an X -flap. (X is the set of vertices currently occupied by the cops and R tells us where the robber is — since she can run arbitrarily fast, all that matters is which component of $G - X$ contains her.) At the start, the cops choose a subset X_0 , and the robber chooses an X_0 -flap R_0 . Note that if there are k cops in the game, then $|X_0| \leq k$. At the start of round i , we have some position (X_{i-1}, R_{i-1}) . The cops choose a new set $X_i \subseteq V(G)$ with $|X_i| \leq k$ (and no other restriction), and announce it. Then the robber, knowing X_i , chooses an X_i -flap R_i which touches R_{i-1} . If this is not possible then the cops have won. Otherwise, i.e. if the robber never runs out of valid moves, the robber wins.

The following lemma establishes a link between the two games.

Lemma 4.2. *Let G be a graph. If k cops can capture a robber with unbounded speed in the Cops and Robber game in G , then $k(\Delta + 1)$ cops can capture the robber in the Helicopter Cops and Robber game in G .*

Proof. We consider two games played in two copies of G : the first one, which we call the *real game*, is a game of Helicopter Cops and Robber with $k(\Delta + 1)$ cops; and the second one, the *virtual game*, is the usual Cops and Robber game with k cops and a robber with unbounded speed. Given a winning strategy for the cops in the virtual game, we need to give a capturing strategy for the cops in the real game. We translate the moves of the cops from the virtual game to the real game, and translate the moves of the robber from the real game to the virtual game, in such a way that all the translated moves are valid, and if the robber is captured in the virtual game, then she is captured in the real game as well. Hence, as the cops have a winning strategy in the virtual game, they have a winning strategy in the real game, too.

In the virtual game, initially the cops choose a subset C_0 of vertices. Then the real cops choose $X_0 = \overline{N}(C_0)$. Recall that $|C_0| \leq k$, so $|X_0| \leq k(\Delta + 1)$. The real robber chooses R_0 , which is an X_0 -flap, and the virtual robber chooses an arbitrary vertex $r_0 \in R_0$. In general, at the end of round $i - 1$ we have $X_{i-1} = \overline{N}(C_{i-1})$ and $r_{i-1} \in R_{i-1}$.

Suppose the virtual robber is not captured in round i . In round i , first the virtual cops move to a new set C_i . Each cop either stays still or moves to a neighbour, thus $C_i \subseteq \overline{N}(C_{i-1}) = X_{i-1}$ and since R_{i-1} was an X_{i-1} -flap, $C_i \cap R_{i-1} = \emptyset$. The real cops choose $X_i = \overline{N}(C_i)$ and announce it. The real robber, knowing X_i , chooses an X_i -flap R_i that touches R_{i-1} . If she cannot find a valid move then she is captured and the lemma is proved. Otherwise, note that by definition $C_i \cap R_i = \emptyset$. Let r_i be an arbitrary vertex of R_i . The virtual robber moves from r_{i-1} to r_i . Since R_{i-1} and R_i touch, and both of them are connected, $R_{i-1} \cup R_i$ is connected. Moreover, C_i does not intersect $R_{i-1} \cup R_i$, so this is a valid move in the virtual game.

Now, suppose the virtual robber is captured in round i . We claim that if this happens then the real robber has already been captured in one of the previous rounds. If this is not the case, then in round i , the virtual cops move to a new set C_i such that $r_{i-1} \in C_i$. Each cop either stays still or moves to a neighbour, thus $C_i \subseteq \overline{N}(C_{i-1}) = X_{i-1}$ and since R_{i-1} was an X_{i-1} -flap, $C_i \cap R_{i-1} = \emptyset$. But $r_{i-1} \in C_i$ because the virtual robber has been captured in round i , and $r_{i-1} \in R_{i-1}$, thus $r_{i-1} \in C_i \cap R_{i-1}$, which is a contradiction. This shows that the real robber will be captured even before the virtual robber, and the proof is complete. ■

Seymour and Thomas [26] proved the following theorem.

Theorem 4.3 ([26]). *The minimum number of cops needed to capture a robber in Helicopter Cops and Robber game is equal to the treewidth of the graph plus one.*

Using this we have the following.

Theorem 4.4. *For every graph G we have*

$$\frac{tw(G) + 1}{\Delta(G) + 1} \leq c_\infty(G) \leq tw(G) + 1,$$

and these bounds are tight.

Proof. The lower bound follows from Lemma 4.2 and Theorem 4.3. To prove tightness of the lower bound, let G be the complete graph on n vertices. Then it follows from part (a) of Proposition 4.1 that G has treewidth $n - 1$. A single cop can capture the robber in G , since G has domination number one. Hence, the complete graph on n vertices has treewidth $n - 1$, maximum degree $n - 1$, and cop number 1, so the lower bound is tight.

Now we prove the upper bound. Consider a tree decomposition (T, W) of G having minimum width. Assume that there are $tw(G) + 1$ cops in the game, so for every $t \in V(T)$, there are at least $|W_t|$ cops in the game. The cops start at W_{t_1} for some arbitrary $t_1 \in V(T)$. Assume that the robber starts at r_0 , and let t be such that $r_0 \in W_t$. Let t_2 be the neighbour of t_1 in the unique (t_1, t) -path in T . Let T_1 and T_2 be the components of $T - t_1 t_2$, with $t_1 \in T_1$ and $t_2 \in T_2$. Define $X = W_{t_1} \cap W_{t_2}$, $U_1 = \cup_{t \in T_1} W_t$, and $U_2 = \cup_{t \in T_2} W_t$. So the cops are all in U_1 and the robber is at a vertex in $U_2 \setminus X$. Note that the number of cops is at least $|W_{t_2}|$. Now the cops move in order to occupy W_{t_2} , in such a way that the cops in X stay still. After some rounds, the cops will be located at W_{t_2} , and during those rounds the robber could not escape from $U_2 \setminus X$, because by part (b) of Proposition 4.1, there is no edge between $U_1 \setminus X$ and $U_2 \setminus X$. When the cops have established in W_{t_2} , the total space available to the robber has been decreased. Continuing similarly the cops will eventually capture the robber.

Next we prove that the upper bound is tight. Let $m \geq 4$ be a positive integer. Define graph G as follows. G has a total of $m + 2m\binom{m}{2}$ vertices, with a certain independent set $\{v_1, \dots, v_m\}$, such that every two of the v_i 's are connected by m disjoint paths of length 3, and G does not have any other edge. Thus G has a total of $3m\binom{m}{2}$ edges. We show that there exists a tree decomposition of G with width $\max\{m - 1, 3\}$. Let T be the star with $1 + m\binom{m}{2}$ vertices, and let r be its dominating vertex. Define $W_r = \{v_1, \dots, v_m\}$.

To each path $v_i u_1 u_2 v_j$ assign a leaf l of the tree and set $W_l = \{v_i, u_1, u_2, v_j\}$. It is easy to verify that (T, W) is a tree decomposition of G with width $\max\{m - 1, 3\}$. Note that $m \geq 4$, so $tw(G) \leq m - 1$.

Now we show that $c_\infty(G) \geq m$, which completes the proof. It suffices to show that $m - 1$ cops cannot capture a robber with unbounded speed. Say a cop *controls* a vertex u if the cop is at u or at an adjacent vertex. If there are $m - 1$ cops in the game, we show that the robber can play such that at the end of each round, if the cops are in $C \subseteq V(G)$, then the robber is at a vertex $r \in \{v_1, \dots, v_m\} \setminus \overline{N}(C)$. The robber can choose such a vertex at the beginning, because the distance between any two of the v_i 's is 3, so each cop can control at most one of the v_i 's. Assume that at the end of round i the cops are in C_i and the robber is at $r_i \in \{v_1, \dots, v_m\} \setminus \overline{N}(C_i)$. In round $i + 1$, first the cops move to $C_{i+1} \subseteq \overline{N}(C_i)$. So the robber is not captured. There exists a vertex $r_{i+1} \in \{v_1, \dots, v_m\} \setminus \overline{N}(C_{i+1})$, because every cop controls at most one of the v_i 's. If $r_{i+1} = r_i$ then the robber does not move at all. Otherwise, there are m disjoint (r_i, r_{i+1}) -paths in G , and $m - 1$ cops, so at least one of these paths is cop-free, and the robber moves along that path to r_{i+1} . ■

5 Planar Graphs

In one of the first papers on the original Cops and Robber game, Aigner and Fromme [1] proved that three cops can capture the robber in any planar graph. In this section, we show that if G is planar then $c_\infty(G) = \Theta(tw(G))$. This proves that every planar graph G has $c_\infty(G) = O(\sqrt{n})$, and also gives an $O(1)$ -approximation algorithm for finding the cop number of a planar graph. These results hold also when G does not contain any fixed apex graph as a minor.

An *apex graph* is a graph H that has a vertex v such that $H - v$ is planar. For example, K_5 is an apex graph. The following theorem was proved in a weaker form by Demaine, Fomin, Hajiaghayi, and Thilikos [6], and then in its current form by Demaine and Hajiaghayi [7].

Theorem 5.1 ([6, 7]). *Let H be a fixed apex graph. There is a constant C_H such that the following holds. Let $g : \mathbb{N} \rightarrow \mathbb{N}$ be a strictly increasing function, and $P(G)$ be a graph parameter with the following two properties.*

1. *If G is the $r \times r$ grid augmented with additional edges such that each vertex is incident to C_H edges to non-boundary vertices of the grid, then $P(G) \geq g(r)$.*

2. $P(G)$ does not increase by contracting an edge of G .

Then, for any graph G that does not contain H as a minor, the treewidth of G is $O(g^{-1}(P(G)))$.

Theorem 5.2. *Let H be a fixed apex graph. Any graph G that does not contain H as a minor has $c_\infty(G) = \Theta(\text{tw}(G))$. In particular, if G is planar then $c_\infty(G) = \Theta(\text{tw}(G))$.*

Proof. We show that the parameter $c_\infty(G)$ satisfies the two properties given in Theorem 5.1, with $g(r) = (r + 1)/(5 + C_H)$. First, an augmented $r \times r$ grid has treewidth r and maximum degree at most $4 + C_H$, so by Theorem 4.4 its cop number is at least $(r + 1)/(5 + C_H)$.

Second, we need to show that the cop number does not increase by contracting an edge. It is not difficult to show that contracting an edge does not help the robber, since she has unbounded speed, and it does not hurt the cops. Therefore, contracting an edge does not increase the cop number.

Therefore, by Theorem 5.1, if G does not contain H as a minor, then $\text{tw}(G) = O(c_\infty(G))$. By Theorem 4.4, $c_\infty(G) \leq \text{tw}(G) + 1$, so we have $c_\infty(G) = \Theta(\text{tw}(G))$. To get the second statement, note that a planar graph does not contain K_5 as a minor. ■

Corollary 5.3. *Let H be a fixed apex graph. Any graph G that does not contain H as a minor has $c_\infty(G) = O(\sqrt{n})$, and this bound is tight. In particular, any planar graph G has $c_\infty(G) = O(\sqrt{n})$, and this bound is tight.*

Proof. It is known (see, e.g., [2]) that if G does not have H as a minor, then $\text{tw}(G) = O(\sqrt{n})$. The $m \times m$ grid has m^2 vertices and by Theorem 4.4, its cop number is at least $(m + 1)/5$. Hence the bound is tight. ■

Corollary 5.4. *Let H be a fixed apex graph. There is a constant-factor approximation algorithm for computing the cop number of a graph that does not contain H as a minor. In particular, There is a constant-factor approximation algorithm for computing the cop number of a planar graph.*

Proof. Feige, Hajiaghayi, and Lee [10] have developed an $O(1)$ -approximation algorithm for finding the treewidth of a graph that does not contain H as a minor. ■

6 Cartesian Products of Complete Graphs, and Hypercube Graphs

Let G_1, G_2, \dots, G_m be graphs. Define G to be the graph with vertex set $V(G_1) \times V(G_2) \times \dots \times V(G_m)$ with vertices (u_1, u_2, \dots, u_m) and (v_1, v_2, \dots, v_m) being adjacent if there exists an index $1 \leq j \leq m$ such that

- $u_i = v_i$ for all $i \neq j$, and
- u_j and v_j are adjacent in G_j .

Then G is called the *Cartesian product* of G_1, G_2, \dots, G_m . If every G_i is isomorphic to an edge, then the graph G is called the *m-hypercube* graph and denoted by \mathcal{H}_m . In this section we give bounds for the Cartesian product of complete graphs with the same size, and tighter bounds for hypercube graphs. Neufeld and Nowakowski [21] have studied the original Cops and Robber game played on products of graphs. They have determined exactly the number of cops needed to capture the robber, when G is the Cartesian product of complete graphs with not necessarily the same size, and when G is the Cartesian product of an arbitrary number of trees and cycles.

First, we prove an easy lemma, which gives a weak upper bound for the cop number of the Cartesian product of graphs.

Lemma 6.1. *Let G_1, G_2, \dots, G_m be graphs and let n_i denote the number of vertices of G_i for $1 \leq i \leq m$. Let G be the Cartesian product of G_1, G_2, \dots, G_m , and $n = |V(G)| = n_1 n_2 \dots n_m$. Then we have*

$$c_\infty(G) \leq \frac{nc_\infty(G_1)}{n_1}.$$

Proof. We give a strategy for $nc_\infty(G_1)/n_1$ cops to capture the robber in G . Let $k = c_\infty(G_1)$. By definition, there is a winning strategy for k cops when the game is played in G_1 . We consider a *virtual game*, in which k virtual cops are capturing a virtual robber in G_1 . (Using a virtual game for bounding the cop number is also used in the proof of Lemma 4.2, where it has been explained in more detail.) For every virtual cop, we put $n/n_1 = n_2 n_3 \dots n_m$ real cops in the real game, such that if the virtual cop is in $u_1 \in V(G_1)$, then the real cops occupy $\{u_1\} \times V(G_2) \times \dots \times V(G_m)$. Also, if the real robber is at $(v_1, \dots, v_m) \in G$, then the virtual robber is at $v_1 \in G_1$. It is not hard to see that the real cops can move in such a way that these constraints hold throughout the games. Hence,

once the virtual robber has been captured, the real robber has also been captured, and the proof is complete. \blacksquare

Theorem 6.2. *Let G_1, G_2, \dots, G_m be graphs, and let G be the Cartesian product of G_1, G_2, \dots, G_m , and $n = |V(G)|$. Then we have*

- (a) *There exist positive constants κ_1, κ_2 such that if every G_i is isomorphic to the complete graph on k vertices, then*

$$\frac{\kappa_1 n}{km\sqrt{m}} \leq c_\infty(G) \leq \min \left\{ \frac{n}{k}, \frac{\kappa_2 n}{\sqrt{m}} \right\}.$$

- (b) *If every G_i is isomorphic to an edge, i.e. if G is the m -hypercube \mathcal{H}_m , then there exist constants $\eta_1, \eta_2 > 0$ such that*

$$\frac{\eta_1 n}{m\sqrt{m}} \leq c_\infty(G) \leq \frac{\eta_2 n}{m}.$$

Proof. (a) Sunil Chandran and Kavitha [4] have proved that

$$tw(G) = \Theta \left(\frac{n}{\sqrt{m}} \right).$$

As G has maximum degree $O(mk)$, the lower bound follows from Theorem 4.4. The upper bound $c_\infty(G) = O(n/\sqrt{m})$ follows from the same theorem, and the bound $c_\infty(G) \leq n/k$ follows from Lemma 6.1, since G_1 is a complete graph and has $c_\infty(G_1) = 1$.

- (b) We claim that for any positive m , the m -hypercube \mathcal{H}_m has domination number at most $2^{m+1}/(m+1)$. Indeed, if for some positive integer k , $m = 2^k - 1$, then it is well known that \mathcal{H}_m has domination number exactly $2^m/(m+1)$ (see [23] for example). Otherwise, let k be the largest integer with $2^k - 1 \leq m$. Thus $m < 2^{k+1} - 1$. It is easy to see that for every graph G with domination number r , the Cartesian product of G and an edge has domination number at most $2r$. Hence one can prove using induction that for $i \geq 2^k - 1$, the domination number of \mathcal{H}_i is at most

$$\frac{2^{2^k-1}}{2^k} 2^{i-(2^k-1)} = 2^{i-k}.$$

In particular, the domination number of \mathcal{H}_m is at most $2^{m-k} < \frac{2^{m+1}}{m+1}$.

The upper bound follows from the above claim (recall that $n = 2^m$), and the fact that the domination number is always an upper bound for the cop number.

Sunil Chandran and Kavitha [4] have proved that $tw(\mathcal{H}_m) = \Theta(2^m/\sqrt{m})$. Since \mathcal{H}_m has maximum degree m , the lower bound follows from Theorem 4.4. \blacksquare

7 Existence of Graphs with Linear Cop Number

Theorem 4.4 is especially useful for giving lower bounds for the cop number, when the graph has small maximum degree. To illustrate this, we use it to give a short proof for the fact that for each n , there exists a connected graph on n vertices with cop number $\Theta(n)$, which is proved by Frieze et al. [14] using other ideas.

Theorem 7.1. *For each n , there exists a connected graph on n vertices with cop number $\Theta(n)$.*

Proof. Let G be an Erdős-Rényi random graph with n vertices and $2n$ edges. Klops [19] has proved that there is a positive constant β such that we have $tw(G) > \beta n$ with probability approaching one, as n goes to infinity.

Each vertex of G has average degree $2|E(G)|/|V(G)| = 4$. Hence by Markov's inequality, the probability that a fixed vertex has degree larger than $16/\beta$ is less than $\beta/4$. By linearity of expectation, the expected number of vertices of degree larger than $16/\beta$ is less than $n\beta/4$. Therefore by Markov's inequality, with probability at least $1/2$, G has at most $n\beta/2$ vertices of degree larger than $16/\beta$.

Consequently, for n large enough, there exists a graph G_n such that

- $tw(G_n) > \beta n$, and
- G_n has at most $n\beta/2$ vertices of degree larger than $16/\beta$.

Let H_n denote the graph obtained from G_n by deleting all vertices of degree larger than $16/\beta$. Deleting each vertex does not decrease treewidth by more than 1. Thus we have

$$|V(H_n)| \leq n, \quad tw(H_n) \geq n\beta/2, \quad \text{and} \quad \Delta(H_n) \leq 16/\beta.$$

By Theorem 4.4,

$$c_\infty(H_n) \geq \frac{tw(H_n) + 1}{\Delta(H_n) + 1} \geq \frac{tw(H_n)}{2\Delta(H_n)},$$

and so

$$\frac{|V(H_n)|}{c_\infty(H_n)} \leq \frac{2|V(H_n)|\Delta(H_n)}{tw(H_n)} \leq 2n \times \frac{16}{\beta} \times \frac{2}{n\beta} = 64/\beta^2 = O(1),$$

completing the proof. ■

8 Open Problems

In this section we present a few open questions and research directions on this variant of the Cops and Robber game.

1. Fomin et al. [11] asked about the complexity of computing $c_\infty(G)$ when G is an interval graph. We proved that this problem is 3-approximable (see Theorem 2.5), but it is still not known if it is NP-hard or not.
2. We proved that there exist chordal graphs G with $c_\infty(G) = \Omega(n/\log n)$ (see Theorem 3.2). Are there chordal graphs G with $c_\infty(G) = \Omega(n)$?
3. Let H be a fixed apex graph. In Theorem 5.2 we proved that if G does not have H as a minor, then $c_\infty(G) = \Theta(tw(G))$. Is this result true when H is a general graph?
4. In part (b) of Theorem 6.2, we have determined the cop number of the m -hypercube graph up to an $O(\sqrt{m})$ factor. What is the exact value?
5. Fomin et al. [11] proved that computing $c_\infty(G)$ is NP-hard. Is this problem in NP? To show that this problem is in NP, one needs to prove that there is always an efficient way to describe the cops' strategy. This has been done for the Helicopter Cops and Robber game [26].

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